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INVESTIGATION OF CERTAIN OPTIMAL SYSTEMS BY THE AVERAGING METHOD

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We construct the canonic averaging scheme for solving certain optimal control problems on the basis of Pontriagin maximum principle. We assume that the plant is described by a system with rotating phase [1], while the control enters only into the perturbing terms [2]. The analysis is carried out on a large time interval so that the controlled quantities vary significantly. The procedure developed is illustrated by concrete examples of quasi-linear oscillatory systems. The small parameter method for the approximate solution of optimal control problems was employed in [2 – 5].

1. Statement of the problem. We formulate the problem of controlling a certain mechanical plant by small control actions. Let the corresponding system of equations have the form

$$x' = \varepsilon X(\tau, x, y, u, \varepsilon), \quad \tau = \varepsilon(t - t_0) + \tau_0, \quad x(t_0) = x_0 \qquad (1.1)$$
$$y' = Y_0(\tau, x, y) + \varepsilon Y(\tau, x, y, u, \varepsilon), \qquad y(t_0) = y_0$$

Here x, X are n-dimensional vectors; y, Y_0, Y are m-dimensional vectors; u is the l-dimensional control, τ is "slow time", ε is a small scalar parameter, $\varepsilon \in [0, \varepsilon_0]$. We assume that the right-hand sides of system (1.1) have been defined in some, possibly unbounded, region of variation of their arguments and in it satisfy all the necessary smoothness and periodicity conditions which follow from the subsequent constructions. The control's performance criterion will be introduced somewhat later, after the derivation of a standard system with rotating phase.

From (1.1) it follows that when $\varepsilon = 0$ the system becomes uncontrollable

$$y^{\circ\cdot} = Y_0(\tau, x^\circ, y^\circ), \quad \tau = \tau_0 = \text{const}, \quad x^\circ = \text{const}$$

Let this unperturbed system allow a complete $(n + m + 1)$-parameter family of solutions of the form [6]

$$y^\circ = (\Pi/2\pi)[\nu(\tau)(t - t_0) + \psi_0] + \varphi(\nu(\tau)(t - t_0) + \psi_0, \tau, c, x^\circ)$$

Here Π is a constant m-vector with components equal to zero for the oscillating variables; $\nu(\tau)$ is the scalar natural frequency depending on a parameter τ; $\psi = \nu(\tau)(t - t_0) + \psi_0$ is the unperturbed phase, ψ_0 is the phase constant, c is a $(m - 1)$-dimensional vector of the family's independent parameters, $\varphi(\psi, \tau, c, x^\circ)$ is a uniform almost-periodic function of phase ψ. It is natural to assume that the right-hand side of system (1.1) is a uniform almost-periodic function of the rotating components of the fast vector y.

Let us introduce the control's performance criterion for system (1.1). Suppose that the purpose of control is to minimize the functional

$$J = g(x(T), c(T), \psi(T)) + \varepsilon \int_{t_0}^{T} G(\tau, x, c, \psi, u, \varepsilon)\, dt \qquad (1.2)$$

where, by virtue of the perturbed system (1.1), the parameter c is a slowly varying function of time t. It is natural to require that the integrand G be uniformly almost periodic with respect to the fast variable (phase) ψ and be analogously smooth. The integral's upper limit is taken to be a fixed quantity; moreover, T is of the order of ε^{-1} so that the controlled slow variables x and c vary significantly on the time interval $[t_0, T]$ being considered, i.e. by a magnitude of the order of unity. For such a choice of T the factor ε in front of the integral in (1.2) is chosen for the purpose of equalizing the orders of the quantities occurring in the functional and such that the resulting value of the optimal control $u(t)$ is of the order of unity in magnitude.

For convenience of treatment we reduce system (1.1) to a standard form with rotating phase [1, 7]

$$x^\cdot = \varepsilon X(\tau, x, y^\circ(\psi, \tau, c, x), u, \varepsilon)$$
$$c^\cdot = \varepsilon C(\tau, x, \psi, c, u, \varepsilon)$$
$$\psi^\cdot = \nu(\tau) + \varepsilon\Psi(\tau, x, \psi, c, u, \varepsilon)$$

Here the functions C and Ψ are determined as the solution of the linear algebraic system

$$\frac{\partial y^\circ}{\partial \psi}\Psi + \frac{\partial y^\circ}{\partial c}C = Y(\tau, x, y^\circ, u, \varepsilon) - \frac{\partial y^\circ}{\partial x}X(\tau, x, y^\circ, u, \varepsilon) - \frac{\partial y^\circ}{\partial \tau}$$

where the differentiation of ψ with respect to τ is not carried out. It is assumed that the functional determinant (the Wronskian for the unperturbed variational system) $\det(\partial y^\circ / \partial\psi, \partial y^\circ / \partial c)$ is not zero in the considered region of the arguments τ, ψ, c and x.

Uniting x and c into one slow vector a, we write the system with rotating phase obtained in the standard form

$$a^\cdot = \varepsilon f(\tau, a, \psi, u, \varepsilon), \quad a(t_0) = a_0 \qquad (1.3)$$
$$\psi^\cdot = \nu(\tau) + \varepsilon F(\tau, a, \psi, u, \varepsilon), \quad \psi(t_0) = \psi_0$$

Here a is a slow vector of arbitrary dimension n, ψ is the scalar rotating phase, i.e.

$\nu(\tau) \geqslant \nu_0 > 0$; the initial values of these variables are defined in terms of x_0, y_0. Note that when the above-stated assumptions are satisfied, the right-hand sides of system (1.3) are smooth functions of their arguments in the region being considered and are uniform and almost-periodic in ψ.

The control's performance criterion (1.2) can be rewritten as

$$J = g(a(T),\ \psi(T),\ \varepsilon) + \varepsilon \int_{t_0}^{T} G(\tau, a, \psi, u, \varepsilon)\, dt \to \min \qquad (1.4)$$

with respect to u, where $u(t) \in U$, while U is some convex region of admissible values of the control l-vector. Note that without loss of generality we can set $G \equiv 0$. As a matter of fact, having augmented the vector a by one more component, varying in accordance with the equation

$$a'_{n+1} = \varepsilon G(\tau, \psi, u, \varepsilon), \qquad a_{n+1}(t_0) = 0$$

functional (1.4) can be presented as a function of final values of the phase coordinates: $J = g(a(T), \psi(T), \varepsilon) + a_{n+1}(T)$, i.e. we can take it that $J = g(a(T), \psi(t), \varepsilon)$, $a = (a_1, \ldots, a_n, a_{n+1})$.

The paper's aim is to construct an approximate optimal solution of problem (1.3),(1.4) to any degree of accuracy with respect to the small parameter ε on an asymptotically arbitrarily large time interval. The optimization problem is solved on the basis of Pontriagin maximum principle [8] under the assumption that such a solution exists and is unique. We note that a number of practical problems of the control of nonlinear and quasi-linear oscillatory and rotatory systems by small control actions, reduce to optimal problems (1.3),(1.4). A quasi-linear oscillatory system with one degree of freedom is examined in Sect. 3.

2. Asymptotic solution of a two-point problem of the maximum principle. The Hamiltonian for problem (1.3),(1.4) has the form

$$H(\tau, a, \psi, p, q, u, \varepsilon) = \varepsilon(f, p) + (\nu + \varepsilon F)q - \varepsilon G$$

where p is a vector adjoint to a; q is a scalar variable adjoint to ψ. The necessary optimality condition for the control $u = v(t)$ (Pontriagin maximum principle [8]) is that

$$H(\tau, a, \psi, p, q, v, \varepsilon) = \max_{u \in U} H(\tau, a, \psi, p, q, u, \varepsilon)$$

at any instant $t \in [t_0, T]$; the variables a and ψ satisfy system (1.3) with $u = v(t)$, while p and q satisfy the corresponding adjoint system and the transversality conditions at the right end.

Let the control $u = u^*(\tau, a, \psi, p, q, \varepsilon)$, rendering the maximum of the function H when the other arguments are fixed and being a sufficiently smooth function uniform and almost-periodic in ψ, be known and unique. We introduce the notation

$$H^* = H(\tau, a, \psi, p, q, u^*, \varepsilon) = \nu(\tau)q + \varepsilon h(\tau, a, \psi, p, q, u^*, \varepsilon)$$

Then, the solving of optimal problem (1.3),(1.4) reduces to constructing the solution of the two-point problem for the Hamilton system of equations

$$a' = \varepsilon f^*(\tau, a, \psi, p, q, \varepsilon), \quad \psi' = \nu(\tau) + \varepsilon F^*(\tau, a, \psi, p, q, \varepsilon) \qquad (2.1)$$

$$p' = \varepsilon\left[\frac{\partial G}{\partial a} - \frac{\partial (f, p)}{\partial a} - \frac{\partial F}{\partial a} q\right]^*, \quad q' = \varepsilon\left[\frac{\partial G}{\partial \psi} - \left\langle \frac{\partial f}{\partial \psi}, p \right\rangle - \frac{\partial F}{\partial \psi} q\right]^*$$

Here the asterisk in the degree position denotes that the corresponding functions and their partial derivatives are taken with the known expression for $u = u^*(\tau, a, \psi, p, q, \varepsilon)$. The right-end conditions for the adjoint vector p, q, i.e. the transversality conditions, have the form

$$p(T, \varepsilon) = -\left.\frac{\partial g}{\partial a}\right|_T, \qquad q(T, \varepsilon) = -\left.\frac{\partial g}{\partial \psi}\right|_T \tag{2.2}$$

For the sake of generality we can assume that the initial conditions for a, ψ also depend on ε

$$a(t_0) = a_0 = a_0(\varepsilon), \qquad \psi(t_0) = \psi_0 = \psi_0(\varepsilon) \tag{2.3}$$

Suppose that the solution of the maximum principle's boundary-value problem (2.1) – (2.3) has been constructed and is unique. Then the solution of the original optimal problem is known. The optimal control $u = u^*(\tau, a, \psi, p, q, \varepsilon)$ can be determined as the program control by the substitution of the known solution of the boundary-value problem; as the "partial synthesis": $u = u^*(\tau, a, \psi, p(t), q(t), \varepsilon)$ or as the "total synthesis" if we solve the expression $\psi = \psi(t)$ relative to $t = t(\psi)$, which is possible and unique, and we substitute into the adjoint vector. The optimal trajectory is also known, while the minimum value of the functional is computed by quadrature.

Let us apply the method of averaging over the fast phase to system (2.1). As is known [1, 7], the averaging method is connected with a change of variables, defined by certain partial differential relations whose integration leads to arbitrary functions of the slow variables. In [9, 10] it was shown that in the case of the Cauchy's problem for a small t-periodic Hamiltonian of the form $\varepsilon h(x, p, t)$ (x is the coordinate, p is the momentum) these arbitrary functions can be used so that the averaged system also has the canonic form. Below we develop an analogous method of canonic averaging over the fast phase ψ, which simplify significantly the construction of the solution of the boundary-value problem (2.1) – (2.3). The order of the averaged system is lessened by two. Firstly, the slow variables are integrated independently of the phase and, secondly, the mean value of q is constant since the averaged Hamiltonian does not depend upon the phase. If it does not depend on τ, then the "energy" integral is preserved.

We go on to construct an averaged boundary-value problem, simpler to integrate, on the basis of whose solution we construct the approximate solution of the original problem (2.1) – (2.3). Then we make wide use of the fact that the partial derivatives of the Hamiltonian, taken with $u = u^*$, coincide with the full partial derivatives, i.e. with the derivatives of H^* [5]

$$\left.\frac{\partial H}{\partial p}\right|_{u^*} = \frac{\partial H^*}{\partial p}, \quad \left.\frac{\partial H}{\partial q}\right|_{u^*} = \frac{\partial H^*}{\partial q}, \quad \left.\frac{\partial H}{\partial a}\right|_{u^*} = \frac{\partial H^*}{\partial a}, \quad \left.\frac{\partial H}{\partial \psi}\right|_{u^*} = \frac{\partial H^*}{\partial \psi}$$

since there hold the identities

$$\left.\frac{\partial H}{\partial u}\right|_{u^*}\frac{\partial u^*}{\partial p} \equiv 0, \quad \left.\frac{\partial H}{\partial u}\right|_{u^*}\frac{\partial u^*}{\partial q} \equiv 0, \quad \left.\frac{\partial H}{\partial u}\right|_{u^*}\frac{\partial u^*}{\partial a} \equiv 0, \quad \left.\frac{\partial H}{\partial u}\right|_{u^*}\frac{\partial u^*}{\partial \psi} \equiv 0$$

We shall construct a univalent canonic change of variables (a, ψ, p, q) to the new (averaged) variables $(\xi, \varphi, \eta, \beta)$, characterizable by the generating function $S(\tau, a, \psi, \eta, \beta, \varepsilon)$ [11]

$$p = \frac{\partial S}{\partial a}, \quad q = \frac{\partial S}{\partial \psi}, \quad \xi = \frac{\partial S}{\partial \eta}, \quad \varphi = \frac{\partial S}{\partial \beta} \tag{2.4}$$

such that the new (averaged) Hamiltonian is independent of φ. Moreover, the old and the new variables should coincide in the zero approximation (when $\varepsilon = 0$). The

generating function S and the new Hamiltonian K are connected by the following differential relation:

$$\frac{\partial S}{\partial t} + H^*\left(\tau, a, \psi, \frac{\partial S}{\partial a}, \frac{\partial S}{\partial \psi}, \varepsilon\right) = K(\tau, \xi, \eta, \beta, \varepsilon)$$

which, with due regard to the representations

$$S = (a, \eta) + \psi\beta + \varepsilon\sigma(\tau, a, \psi, \eta, \beta, \varepsilon) \tag{2.5}$$
$$K = \nu(\tau)\beta + \varepsilon k(\tau, \xi, \eta, \beta, \varepsilon)$$
$$p = \eta + \varepsilon\frac{\partial\sigma}{\partial a}, \quad q = \beta + \varepsilon\frac{\partial\sigma}{\partial\psi}, \quad \xi = a + \varepsilon\frac{\partial\sigma}{\partial\eta}, \quad \varphi = \psi + \varepsilon\frac{\partial\sigma}{\partial\beta}$$

reduce to the form

$$\nu\frac{\partial\sigma}{\partial\psi} + h\left(\tau, a, \psi, \eta + \varepsilon\frac{\partial\sigma}{\partial a}, \beta + \varepsilon\frac{\partial\sigma}{\partial\psi}, \varepsilon\right) + \varepsilon\frac{\partial\sigma}{\partial\tau} = \tag{2.6}$$
$$k\left(a + \varepsilon\frac{\partial\sigma}{\partial\eta}, \eta, \beta, \varepsilon\right)$$

If function h is piecewise-continuous and uniformly almost-periodic in ψ, continuous in τ, and continuously differentiable sufficiently often in the remaining arguments, then with the aid of these relations the desired functions σ and k can be computed to any degree of accuracy with respect to the small parameter ε in the form

$$\sigma(\tau, a, \psi, \eta, \beta, \varepsilon) = \sigma_0 + \varepsilon\sigma_1 + \varepsilon^2\sigma_2 + \ldots \tag{2.7}$$
$$k(\tau, \xi, \eta, \beta, \varepsilon) = k_0 + \varepsilon k_1 + \varepsilon^2 k_2 + \ldots$$

With due regard to (2.5) we substitute series (2.7) into Eq. (2.6) and we equate the coefficients of like powers of ε. We obtain a linked sequence of equations and, in particular, the relation

$$\nu(\tau)\frac{\partial\sigma_0}{\partial\psi} + h(\tau, a, \psi, \eta, \beta, 0) = k_0(\tau, a, \eta, \beta)$$

whose solution is:

$$k_0(\tau, a, \eta, \beta) = \langle h_0\rangle \equiv \lim_{N\to\infty}\frac{1}{N}\int_{\psi_0}^{N} h(\tau, a, \psi, \eta, \beta, 0)\,d\psi \tag{2.8}$$
$$\sigma_0(\tau, a, \psi, \eta, \beta) = -\frac{1}{\nu}\int(h_0 - \langle h_0\rangle)\,d\psi$$

For the succeeding unknown coefficients k_i, σ_i $(i \geqslant 1)$ there hold the analogous expressions

$$k_i = \langle h_i\rangle, \qquad \sigma_i = -\frac{1}{\nu}\int(h_i - \langle h_i\rangle)\,d\psi \tag{2.9}$$

in which the functions h_i are computed on the basis of the known quantities. For example, for $i = 1, 2$

$$h_1(\tau, a, \psi, \eta, \beta) = \frac{\partial\sigma_0}{\partial\tau} + \frac{\partial h_0}{\partial\eta}\frac{\partial\sigma_0}{\partial a} + \frac{\partial h_0}{\partial\beta}\frac{\partial\sigma_0}{\partial\psi} + \left(\frac{\partial h}{\partial\varepsilon}\right)_0 - \frac{\partial k_0}{\partial a}\frac{\partial\sigma_0}{\partial\eta} \tag{2.10}$$

$$h_2(\tau, a, \psi, \eta, \beta) = \frac{\partial\sigma_1}{\partial\tau} + \frac{\partial h_0}{\partial\eta}\frac{\partial\sigma_1}{\partial a} + \frac{1}{2}\frac{\partial^2 h_0}{\partial\eta^2}\left(\frac{\partial\sigma_0}{\partial a}\right)^2 + \frac{\partial h_0}{\partial\beta}\frac{\partial\sigma_1}{\partial\psi} +$$
$$\frac{1}{2}\frac{\partial^2 h_0}{\partial\beta^2}\left(\frac{\partial\sigma_0}{\partial\psi}\right)^2 + \frac{\partial^2 h_0}{\partial\eta\partial\beta}\frac{\partial\sigma_0}{\partial a}\frac{\partial\sigma_0}{\partial\psi} + \frac{\partial^2 h_0}{\partial\eta\partial\varepsilon}\frac{\partial\sigma_0}{\partial a} + \frac{\partial^2 h_0}{\partial\beta\partial\varepsilon}\frac{\partial\sigma_0}{\partial\psi} + \frac{1}{2}\left(\frac{\partial^2 h}{\partial\varepsilon^2}\right)_0 -$$
$$\frac{\partial k_0}{\partial a}\frac{\partial\sigma_1}{\partial\eta} - \frac{1}{2}\frac{\partial^2 k_0}{\partial a^2}\left(\frac{\partial\sigma_0}{\partial\eta}\right)^2 - \frac{\partial k_1}{\partial a}\frac{\partial\sigma_0}{\partial\eta}$$

and so on. Obviously, the accuracy of the constructions carried out is restricted by the degree of smoothness of the original Hamiltonian. If u^*, the function G, and the right-hand sides of system (2.1) are only piecewise-continuous, then we can write out the so-called first-approximation equations

$$\xi^{\cdot} = \frac{\partial K}{\partial \eta} = \varepsilon \langle f_0(\tau, \xi, \eta, \beta) \rangle \tag{2.11}$$

$$\eta^{\cdot} = -\frac{\partial K}{\partial \xi} = \varepsilon \left\langle \left[\frac{\partial G_0}{\partial \xi} - \frac{\partial (f_0, \eta)}{\partial \xi} - \frac{\partial F_0}{\partial \xi} \beta \right]^* \right\rangle$$

$$\beta^{\cdot} = -\frac{\partial K}{\partial \varphi} = 0, \quad \varphi^{\cdot} = \frac{\partial K}{\partial \beta} = \nu(\tau) + \varepsilon \langle F_0(\tau, \xi, \eta, \beta) \rangle$$

Here the zero in the subscipt denotes that ε is assumed equal to zero, while the angle brackets denote averaging over the phase. In comparison with the exact Eqs. (2.1) system (2.11) yields an error of the order of ε on the interval $[t_0, \ T]$, $T \sim \varepsilon^{-1}$. Therefore, the initial and boundary conditions should be written out to this same accuracy

$$\xi_0 = a_0(0), \quad \varphi_0 = \psi_0(0), \quad \eta(T) = -\frac{\partial g_0}{\partial \xi}\Big|_T, \quad \beta = -\frac{\partial g_0}{\partial \psi}\Big|_T \tag{2.12}$$

From the penultimate equation in (2.11) it follows that $\beta = \text{const}$, i.e. β occurs everywhere as a parameter. We note further that if $\langle h_0 \rangle$ is independent of τ, then system (2.11) admits the integral $k(\xi, \ \eta, \ \beta) = \text{const}$, which makes it possible to lessen the system's order. In the case of a system with one degree of freedom the problem can be solved in quadratures. If the slow variables have been computed, then the value of $\varphi(t)$ also is obtained by quadrature.

The so-called improved first approximation [7], i.e. one satisfying system (2.1) with an error $\sim \varepsilon^2$ is

$$p = \eta + \varepsilon \frac{\partial \sigma_0}{\partial \xi}, \qquad q = \beta + \varepsilon \frac{\partial \sigma_0}{\partial \varphi} \tag{2.13}$$

$$a = \xi - \varepsilon \frac{\partial \sigma_0}{\partial \eta}, \quad \psi = \varphi - \varepsilon \frac{\partial \sigma_0}{\partial \beta} \quad (\sigma_0 = \sigma_0(\tau, \xi, \varphi, \eta, \beta))$$

Thus, suppose that the necessary number of coefficients σ_i, k_i have been computed by formulas (2.8) – (2.10), i.e. the generating function and the averaged Hamiltonian (2.5) have been computed to the necessary degree of accuracy with respect to the small parameter ε. Then the averaged canonic system has a simpler form than the original (2.1)

$$\xi^{\cdot} = \frac{\partial K}{\partial \eta} = \varepsilon \frac{\partial k}{\partial \eta}, \qquad \eta^{\cdot} = -\frac{\partial K}{\partial \xi} = -\varepsilon \frac{\partial k}{\partial \xi} \tag{2.14}$$

$$\varphi = \frac{\partial K}{\partial \beta} = \nu(\tau) + \varepsilon \frac{\partial k}{\partial \beta}, \qquad \beta = \text{const}$$

and determines the averaged solution to the accuracy needed on the whole interval $[t_0, \ T]$ (when $T \sim \varepsilon^{-1}$ this accuracy is lower by unity than the accuracy of computation of the averaged Hamiltonian).

Let us now find the initial and boundary conditions for the averaged variables. Suppose that the general solution of system (2.14) has been constructed in the form

$$\xi = \xi(\tau, \ \varepsilon t, \ \xi_0, \ \eta_T, \ \beta, \ \varepsilon), \qquad \xi \mid_{t_0} = \xi_0 \tag{2.15}$$

$$\eta = \eta(\tau, \ \varepsilon t, \ \xi_0, \ \eta_T, \ \beta, \ \varepsilon), \qquad \eta \mid_T = \eta_T$$

$$\varphi = \varphi_0 + \int_{t_0}^{t} \left[\nu(\tau_1) + \varepsilon \frac{\partial k}{\partial \beta}\right] dt_1, \quad \tau_1 = \varepsilon(t_1 - t_0) + \tau_0, \quad \beta = \text{const}$$

Here the parameters ξ_0, η_T, β and φ_0 as yet unknown, are subject to determination. We derive the relations needed. For this purpose the last two equations in (2.5)

$$\xi = \alpha + \varepsilon \frac{\partial \sigma}{\partial \eta}, \qquad \varphi = \psi + \varepsilon \frac{\partial \sigma}{\partial \beta}$$

are solved with respect to a and ψ

$$a = \xi + \varepsilon A(\tau, \xi, \varphi, \eta, \beta, \varepsilon), \quad \psi = \varphi + \varepsilon \Psi(\tau, \xi, \varphi, \eta, \beta, \varepsilon) \quad (2.16)$$

The result obtained is substituted into the expressions for the adjoint variables p and q

$$p = \eta + \varepsilon P(\tau, \xi, \varphi, \eta, \beta, \varepsilon), \quad q = \beta + \varepsilon Q(\tau, \xi, \varphi, \eta, \beta, \varepsilon) \quad (2.17)$$

where P and Q are functions of $\partial\sigma / \partial a$ and $\partial\sigma / \partial\psi$ into which the found expressions (2.16) have been substituted. We now make use of the initial and boundary conditions (2.3), (2.2)

$$\begin{aligned} a_0 &= \xi_0 + \varepsilon A(\tau_0, \xi_0, \varphi_0, \eta(\varepsilon t_0), \beta, \varepsilon) \\ \psi_0 &= \varphi_0 + \varepsilon \Psi(\tau_0, \xi_0, \varphi_0, \eta(\varepsilon t_0), \beta, \varepsilon) \\ -\frac{\partial g}{\partial a}\Big|_T &= \eta_T + \varepsilon P(\tau_T, \xi(\varepsilon T), \varphi(T), \eta_T, \beta, \varepsilon) \\ -\frac{\partial g}{\partial \psi}\Big|_T &= \beta + \varepsilon Q(\tau_T, \xi(\varepsilon T), \varphi(T), \eta_T, \beta, \varepsilon) \end{aligned} \quad (2.18)$$

The system of equations obtained is solved relative to the unknown parameters ξ_0, φ_0, η_T, β. Obviously, all these computations should be carried out to the accuracy needed.

We note that the approximate construction of expressions (2.16), as well as the computation of the desired parameters can be carried out by series expansion or by successive approximations in powers of the small parameter ε. For example, the successive approximations scheme for the determination of expressions (2.16) has the form

$$a^{(i+1)} = \xi - \varepsilon \frac{\partial}{\partial \eta} \sigma(\tau, a^{(i)}, \psi^{(i)}, \eta, \beta, \varepsilon), \quad a^{(0)} = \xi$$

$$\psi^{(i+1)} = \varphi - \varepsilon \frac{\partial}{\partial \beta} \sigma(\tau, a^{(i)}, \psi^{(i)}, \eta, \beta, \varepsilon), \quad \psi^{(0)} = \varphi, \quad i = 1, 2, \ldots$$

The unknown parameters are determined by a similar method

$$\begin{aligned} \xi_0^{(l+1)} &= a_0 - \varepsilon A(\tau_0, \xi_0^{(l)}, \varphi_0^{(l)}, \eta^{(l)}(\varepsilon t_0), \beta^{(l)}, \varepsilon) \\ \varphi_0^{(l+1)} &= \psi_0 - \varepsilon \Psi(\tau_0, \xi_0^{(l)}, \varphi_0^{(l)}, \eta^{(l)}(\varepsilon t_0), \beta^{(l)}, \varepsilon) \\ \eta_T^{(l+1)} &= -\frac{\partial g_{l+1}}{\partial a}\Big|_T - \varepsilon P(\tau_T, \xi^{(l)}(\varepsilon T), \varphi^{(l)}(T), \eta_T^{(l)}, \beta^{(l)}, \varepsilon) \\ \beta^{(l+1)} &= -\frac{\partial g_{l+1}}{\partial \psi}\Big|_T - \varepsilon Q(\tau_T, \xi^{(l)}(\varepsilon T), \varphi^{(l)}(T), \eta_T^{(l)}, \beta^{(l)}, \varepsilon) \end{aligned}$$

As the zero approximation of the desired parameters we can take the values determined by Eqs. (2.12). For sufficiently small values of ε and smooth right-hand sides the successive approximations converge uniformly, i.e. can be computed with the degree of

accuracy necessary. Thus, if the solution of the averaged system (2.14) has been constructed numerically or analytically, then the solution of the original two-point problem (2.1) – (2.3) can be found to the same accuracy from formulas (2.16), (2.17), and together with this solution, the approximate solution of optimal problem (1.3), (1.4).

3. Example. In practice we often limit ourselves to constructing the first-approximation solution which yields a qualitative picture of the control process and ensures an error of the order of ε on a large time interval of the order of ε^{-1}. Let us consider a weakly-controlled quasi-linear oscillatory system with one degree of freedom

$$x'' + \nu^2(\tau)\, x = \varepsilon f(\tau, x, x', u), \quad x(t_0) = x_0, \quad x'(t_0) = x_0' \tag{3.1}$$

where τ is slow time, $\nu(\tau)$ is the frequency of the oscillations, constant when $\varepsilon = 0$, u is the scalar control. By the substitution

$$x = a \sin \psi, \qquad x' = a\nu \cos \psi$$

Eq. (3.1) is reduced to the following system:

$$a' = \frac{\varepsilon}{\nu(\tau)} [f(\tau, a \sin \psi, a\nu(\tau)\cos \psi, u) - a\nu'(\tau) \cos \psi] \cos \psi$$

$$\psi' = \nu(\tau) + \frac{\varepsilon}{\nu(\tau)\, a} [a\nu'(\tau) \cos \psi - f(\tau, a \sin \psi, a\nu(\tau) \cos \psi, u)] \sin \psi$$

(the prime denotes a derivative with respect to τ). Let $u(t) \in U$, where U is some convex set; we pose the problem of finding an admissible u such that

$$J = g(a(T), \psi(T)) + \varepsilon \int_{t_0}^{T} G(\tau, a, \psi, u)\, dt = \min_{u \in U}$$

For the particular case of a right-hand side of the form

$$f(\tau, x, x', u) = f_0(\tau, x, x') + d(\tau)\, u, \quad |d(\tau)| < \infty$$

we consider the following subcases:

a) $|u| < \infty$, $\quad J = k \dfrac{a^2(T)}{2} + \varepsilon \displaystyle\int_{t_0}^{T} G_0(\tau)\, u^2\, dt$, $\quad G_0(\tau) > 0$, $\quad k \neq 0$

b) $|u| \leqslant u_0$, $\quad J = \pm \dfrac{a^2(T)}{2}$

We examine the first subcase briefly. The Hamilton's function of the system

$$H = \nu q + \frac{\varepsilon}{\nu}(v + du)\, w - \varepsilon G_0 u^2$$

$$v(\tau, a, \psi) = f_0 - a\nu' \cos \psi, \quad w(a, \psi, p, q) = p \cos \psi - \frac{q}{a} \sin \psi$$

is maximal for $u^* = dw / 2\nu G_0$, i.e.

$$H^* = \nu q + \frac{\varepsilon}{\nu} vw + \frac{\varepsilon}{4} \frac{d^2}{\nu^2 G_0} w^2$$

The original two-point problem is described by the equations

$$a' = \frac{\varepsilon}{\nu} v \cos \psi + \frac{\varepsilon}{2} \frac{d^2}{\nu^2 G_0} w \cos \psi, \qquad a(t_0) = a_0$$

$$\psi' = \nu - \frac{\varepsilon}{\nu a} v \sin \psi - \frac{\varepsilon}{2\nu^2} \frac{d^2}{aG_0} w \sin \psi, \qquad \psi(t_0) = \psi_0$$

$$p^{\cdot} = -\frac{\varepsilon}{\nu}\left(\frac{\partial v}{\partial a} + \frac{d^2}{2\nu G_0}\frac{q}{a^2}\sin\psi\right)w - \frac{\varepsilon}{\nu}\frac{q}{a^2}v\sin\psi, \quad p(T) = -ka(T)$$

$$q^{\cdot} = -\frac{\varepsilon}{\nu}\frac{\partial}{\partial\psi}\left(vw + \frac{d^2}{4\nu G_0}w^2\right), \quad q(T) = 0$$

$$(a_0 = (x_0^2 + x_0^{\cdot 2})^{1/2}, \ \psi_0 = \operatorname{arc\,tg}(\nu(\tau_0)\, x_0 / x_0^{\cdot}))$$

The corresponding first-approximation averaged two-point problem is essentially simpler than the original and is described by the equations

$$\xi^{\cdot} = \frac{\varepsilon}{\nu}f_{0c}(\tau, \xi) + \frac{\varepsilon}{4\nu^2}\frac{d^2}{G_0}\eta, \quad \xi(t_0) = a_0$$

$$\eta^{\cdot} = \frac{\varepsilon}{2}\frac{\nu'}{\nu}\eta - \frac{\varepsilon}{\nu}\frac{\partial f_{0c}(\tau, \xi)}{\partial\xi}\eta, \quad \eta(T) = -k\xi(T),$$

$$\varphi^{\cdot} = \nu - \frac{\varepsilon}{\xi}f_{0s}(\tau, \xi), \quad \varphi(t_0) = \psi_0 \quad (\beta = 0)$$

Here

$$\begin{Bmatrix} f_{0c}(\tau, \xi) \\ f_{0s}(\tau, \xi) \end{Bmatrix} = \frac{1}{2\pi}\int_0^{2\pi} f_0(\tau, \xi\sin\omega, \xi\nu(\tau)\cos\psi)\begin{Bmatrix}\cos \\ \sin\end{Bmatrix}\psi\, d\psi$$

If the function f_0 is linear in x and $x^{\cdot}$ or if τ is absent in the system, then the first-approximation equations can be integrated up to the end. For example, let

$$\nu = \text{const}, \ d = 1, \ f_0 = -2\lambda x^{\cdot} + \mu x^3, \ G_0 = 1, \ \lambda, \ \mu = \text{const}$$

Then the approximate solution can be written out explicitly

$$\xi(\varepsilon t) = \left[a_0 - \frac{\eta_T}{8\lambda\nu^2}e^{-\varepsilon\lambda(T-t_0)}\right]e^{-\varepsilon\lambda(t-t_0)} + \frac{\eta_T}{8\lambda\nu^2}e^{\varepsilon\lambda(t-T)}$$

$$\eta(\varepsilon t) = \eta_T e^{\varepsilon\lambda(t-T)}$$

Here

$$\eta_T = -ka_0 e^{-\varepsilon\lambda(T-t_0)}\left\{1 + \frac{k}{8\lambda\nu^2}[1 - e^{-2\varepsilon\lambda(T-t_0)}]\right\}^{-1}$$

$$\lim_{k\to\infty}\eta_T = -8\lambda\nu^2 a_0 e^{-\varepsilon\lambda(T-t_0)}[1 - e^{-2\varepsilon\lambda(T-t_0)}]^{-1}$$

At the end of the control interval the approximate value of the oscillation amplitude equals $\xi(\varepsilon T) = -\eta_T / k$; moreover, $\lim_{k\to\infty}\xi(\varepsilon T) = 0$, and in the first approximation the perturbed oscillation frequency is:

$$\Omega(\varepsilon t) = \nu - \frac{3}{8\nu}\varepsilon\mu\xi^2(\varepsilon t), \quad \varphi(t) = \psi_0 + \int_{t_0}^{t}\Omega(\varepsilon t_1)\, dt_1$$

The optimal control and the functional's minimum value with an error of the order of ε are:

$$u^* = \frac{\eta(\varepsilon t)}{2\nu}\cos\psi \approx v(t) = \frac{\eta_T}{2\nu}e^{\varepsilon\lambda(t-T)}\cos\varphi(t)$$

$$J_{\min} = \frac{\eta_T^2}{2}\left[\frac{1}{k} + \frac{1}{8\lambda\nu^2}(1 - e^{-2\varepsilon\lambda(T-t_0)})\right]$$

We now examine subcase (b) briefly. The Hamiltonian

$$H = \nu q + \frac{\varepsilon}{\nu}(v + du)\, w$$

is maximal for $u^* = u_0 \operatorname{sign} w$ if $w \not\equiv 0$, i.e.

$$H^* = \nu q + \frac{\varepsilon}{\nu}vw + \varepsilon\frac{du_0}{\nu}|w|$$

The original two-point problem is described by the equations

$$a^{\cdot} = \frac{\varepsilon}{\nu} v\cos\psi + \frac{\varepsilon}{\nu} du_0 \cos\psi \operatorname{sign} w, \quad a(t_0) = a_0$$

$$\psi^{\cdot} = \nu - \frac{\varepsilon}{\nu a} v\sin\psi - \frac{\varepsilon}{\nu}\frac{du_0}{a}\sin\psi \operatorname{sign} w, \quad \psi(t_0) = \psi_0$$

$$p^{\cdot} = -\frac{\varepsilon}{\nu}\frac{\partial v}{\partial a} w - \frac{\varepsilon}{\nu}\frac{q}{a^2}\sin\psi\,(v + du^*), \quad p(T) = \mp a(T)$$

$$q^{\cdot} = -\frac{\varepsilon}{\nu}\frac{\partial v}{\partial \psi} w - \frac{\varepsilon}{\nu}\frac{\partial w}{\partial \psi}(v + du^*), \quad q(T) = 0$$

The corresponding first-approximation averaged two-point problem has the form

$$\xi^{\cdot} = \frac{\varepsilon}{\nu}\left(f_{0c}(\tau, \xi) - \frac{\nu'}{2}\xi + \frac{2}{\pi} du_0 \operatorname{sign}\eta\right), \quad \xi(t_0) = a_0$$

$$\eta^{\cdot} = \frac{\varepsilon}{\nu}\left(\frac{\nu'}{2} - \frac{\partial f_{0c}(\tau, \xi)}{\partial \xi}\right)\eta, \quad \eta(T) = \mp\xi(T)$$

$$\varphi^{\cdot} = \nu - \frac{\varepsilon}{\nu}\frac{f_{0s}(\tau, \xi)}{\xi}, \quad \varphi(t_0) = \psi_0$$

Now, under the assumption that η is sign-constant, it is required to construct the solution of the Cauchy problem for the first equation.

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